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SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION THROUGH THE NUMERICAL METHODS

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Abstract: In the present work, we find the solution of one-dimensional wave equation with certain initial and boundary conditions using the numerical methods. The solution is obtained as a polynomial in terms of a dependent variable and time variable.

1. Introduction:

The reduction in the geometric dimension of a wave equation leads to great simplification in the mathematical analysis. Some of the wave equations obtained in elastic media can be reduced to one-dimensional wave equation by assuming the displacements as functions of a single variable apart from time variable t. For such equations one can apply the method described in the present work.

We find the solution of one dimensional wave equation prescribed by initial and boundary conditions as a polynomial in a dependent variable and time variable. In order to determine it, the numerical methods such as double interpolation [1] and Crank – Nicolson method [2] are used.Formulation of the problem

We consider the one-dimensional wave equation

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (1)$$

subject to the boundary conditions

u (0, t) = 0 for t > 0,

u (5, t) = 0 for t > 0,

and initial conditions

u (x, 0) = x (5 – x),

u_t (x, 0) = 0 (5)

where x is the space dimension, t is the time variable, $u_t = \frac{\partial u(x,t)}{\partial t}$ and 0 ≤ x ≤ 5. Solution of the problem

For the function u(x, t) of two variables, let the xt-plane be divided into a lattice of rectangles of length h = 1 and breadth k = ½ by drawing the two families of parallel liner x = mh, y = nk (m = 0, 1, 2, …. and n = 0, 1, 2, ….). The points of intersection of these families of lines are called lattice points. Here we have chosen k such that k = $\frac{h}{2}$ as 2 being the velocity the wave. Thus, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$,, $x_3 = 3$,, $x_4 = 4$,, $x_5 = 5$ and $t_0 = 0$,, $t_1 = 0.5$, $t_2 = 1.0$,, $t_3 = 1.5$, $t_4 = 2.0$, $t_5 = 2.5$.

We denote $u_{rs} = \left(r, \frac{s}{2}\right)$ (r, s = 0,1 2, 3, 4, 5)

The boundary condition (2) gives

$u_{00} = u_{01} = u_{02} = u_{03} = u_{04} = u_{05} = 0$

and the boundary condition (3) gives

$u_{50} = u_{51} = u_{52} = u_{53} = u_{53} = u_{54} = u_{55} = 0$

Now the initial condition (4) gives

$u_{00} = 0$, $u_{10} = 4$, $u_{20} = 6$, $u_{30} = 6$, $u_{40} = 4$, $u_{50} = 0$ (2)

Approximating the partial derivative by the differences, we have (3)

$$u_{i,1} = \frac{1}{2}\left[u_{(i-1)0} + u_{(i+1)0}\right] \quad (4)$$

Hence the initial condition (5) gives

$u_{11} = 3$, $u_{21} = 5$, $u_{31} = 5$, $u_{41} = 3$

The other values at the remaining lattice points are obtained from the recurrence relation

$$u_{i(j+1)} = u_{(i+1)j} + u_{(i-1)j} - u_{i(j-1}$$

and are shown in the Table –1.

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Table - 1

t / x	0	1	2	3	4	5
0	0	4	6	6	4	0
0.5	0	3	5	5	3	0
1.0	0	1	2	2	1	0
1.5	0	-1	-2	-2	-1	0
2.0	0	-3	-5	-5	-3	0
2.5	0	-4	-6	-6	-4	0

The doubles or two – way differences of $u(x,t)$ are defined by

$$\Delta^{m+n}u_{rs} = \Delta_x^{m+0}\Delta_t^{n}u_{rs} = \Delta_t^{0+n}\Delta_x^{m}u_{rs},$$

where

$$\Delta_x u_{rs} = u_{(r+1)s} - u_{rs},$$

and

$$\Delta_t u_{rs} = u_{r(s+1)} - u_{rs}.$$

By considering the u_{0k} (k = 0, 1, 2, 3, 4, 5) values, it can be shown that

$\Delta^{0+k} u_{00} = 0$ (k = 0, 1, 2, 3, 4, 5)

Similarly, by considering u_{5k} (k = 0, 1, 2, 3, 4, 5) we have

$\Delta^{0+k} u_{50} = 0$ (k = 0, 1, 2, 3, 4, 5)

By considering other column values of u(x, t) we have

$\Delta^{0+1}u_{10}$ = -1, $\Delta^{0+2} u_{10}$= -1, $\Delta^{0+3} u_{10}$= 1, $\Delta^{0+4} u_{10}$= -1, $\Delta^{0+5} u_{10}$= 2 (14)

$\Delta^{0+1} u_{20}$ = -1, $\Delta^{0+2} u_{20}$ = 2, $\Delta^{0+3} u_{20}$ = 1, $\Delta^{0+4} u_{20}$ = -1, $\Delta^{0+5} u_{20}$ = - 2 (15)

$\Delta^{0+1} u_{30}$ = -1, $\Delta^{0+2} u_{30}$ = -2, $\Delta^{0+3} u_{30}$ = 1, $\Delta^{0+4} u_{30}$ = 1, $\Delta^{0+5} u_{30}$ = - 2 (16)

$\Delta^{0+1} u_{40}$ = -1, $\Delta^{0+2} u_{40}$ = -1, $\Delta^{0+3} u_{40}$ = 1, $\Delta^{0+4} u_{40}$ = -1 ,$\Delta^{0+5} u_{40}$ = 2 (17)

By considering the u_{k0} (k = 0, 1, 2, 3, 4, 5) values, we have

$\Delta^{1+0} u_{00}$ = 4, $\Delta^{2+0} u_{00}$ = -2, $\Delta^{3+0} u_{0_0}$ = 0 ,$\Delta^{4+0} u_{00}$ = 0, $\Delta^{5+0} u_{00}$ = 0 (18)

Similarly by considering other values, we get

$\Delta^{1+1} u_{01}$ = 3, $\Delta^{2+0} u_{01}$ = -1, $\Delta^{3+0} u_{01}$ = -1,$\Delta^{4+0} u_{01}$ = 1, $\Delta^{5+0} u_{01}$ = 0 (19)

$\Delta^{1+0} u_{02}$ = 1, $\Delta^{2+0} u_{02}$ = 0, $\Delta^{3+0} u_{02}$ = -1,$\Delta^{4+0} u_{02}$ = 1, $\Delta^{5+0} u_{02}$ = 0 (20)

$\Delta^{1+0} u_{03}$ = -1, $\Delta^{2+0} u_{03}$ = 0, $\Delta^{3+0} u_{03}$ = 1,$\Delta^{4+0} u_{03}$ = -1, $\Delta^{5+0} u_{03}$ = 0 (21)

$\Delta^{1+0} u_{04}$ = -3, $\Delta^{2+0} u_{04}$ = 1, $\Delta^{3+0} u_{0_4}$ = 1, $\Delta^{4+0} u_{04}$ = -1, $\Delta^{5+0} u_{04}$ = 0 (22)

$\Delta^{1+0} u_{05}$ = -4, $\Delta^{2+0} u_{05}$ = 2, $\Delta^{3+0} u_{05}$ = 0, $\Delta^{4+0} u_{05}$ = 0, $\Delta^{5+0} u_{05}$ = 0 (23)

The general formula for different order of differences are given by

$$\Delta^{m+n} u_{00} = \Delta^{m+0} u_{00} - n.\,\Delta^{m+0} u_{0(n-1)} + \frac{n(n-1)}{2}.\Delta^{m+0}u_{0(n-2)} + \ldots\ldots\ldots\ldots + (-1)^n \Delta^{m+0}u_{00}$$

$$= \Delta^{0+n}u_{m0} - m\Delta^{0+n}u_{(m-1)0} + \frac{m(m-1)}{2}\Delta^{0+n}u_{(m-2)0}\ldots\ldots + (-1)^n\Delta^{0+n}u_{00} \qquad (24)$$

Using the equations (24) and (12) to (23) we get

$\Delta^{1+1} u_{00}$ = -1 ,

$\Delta^{1+2} u_{00}$ = -1 ,$\Delta^{2+1} u_{00}$ = = 1,

$\Delta^{1+3} u_{00}$ = 1, $\Delta^{2+2} u_{00}$ = 0, $\Delta^{3+1} u_{00}$ = -1 ,

$\Delta^{1+4} u_{00}$ = -1 $\Delta^{2+3} u_{00}$ = -1 ,$\Delta^{3+2} u_{00}$ = 1 $\Delta^{4+1} u_{00}$ (12)

(25)

The formula for double interpolation [2] upto fifth order differences is (13)

$$u(x,t) = u_{00} + \left[\frac{x-x_0}{h}\Delta^{1+0}u_{00} + \frac{t-t_0}{k}\Delta^{0+1}u_{00}\right]$$

$$+\frac{1}{2}\left[\frac{(x-x_0)(x-x_1)}{h^2}\Delta^{2+0}u_{00} + \frac{2(x-x_0)(t-t_0)}{hk}\Delta^{1+1}u_{00} + \frac{(t-t_0)(t-t_1)}{k^2}\Delta^{0+2}u_{00}\right]$$

$$+\frac{1}{3}\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{h^3}\Delta^{3+0}u_{00} + \frac{3(x-x_0)(x-x_1)(t-t_1)}{h^2k}\Delta^{2+0}u_{00} + \frac{3(x-x_0)(t-t_0)(t-t_1)}{hk^2}\Delta^{1+2}u_{00} + \frac{(t-t_0)(t-t_1)(t-t_2)}{k^3}\Delta^{0+3}u_{00}\right]$$

$$\frac{1}{4}\left[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{h^4}\Delta^{4+0}u_{00}\right.$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(t-t_0)}{h^4k}\Delta^{3+1}u_{00}$$

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$$+\frac{6(x-x_0)(x-x_1)(t-t_0)(t-t_1)}{h^2k^2}\Delta^{2+2}u_{00}+$$

$$\frac{4(x-x_0)(t-t_0)(t-t_1)(t-t_2)}{hk^3}\Delta^{1+3}u_{00}$$

$$+\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{k^4}\Delta^{0+1}u_{00}\Bigg]$$

$$+\frac{1}{5}\Bigg[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{h^5}\Delta^{5+0}u_{00}$$

$$+\frac{5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(t-t_0)}{h^4k}\Delta^{4+1}u_{00}$$

$$+\frac{10(x-x_0)(x-x_1)(x-x_2)(t-t_0)(t-t_1)}{h^3k^2}\Delta^{3+2}u_{00}$$

$$+\frac{10(x-x_0)(x-x_1)(t-t_0)(t-t_1)(t-t_2)}{h^2k^3}\Delta^{2+3}u_{00}$$

$$+\frac{5(x-x_0)(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{hk^4}\Delta^{1+4}u_{00}$$

$$+\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_4)}{h^5}\Delta^{5+0}u_{00}\Bigg] \qquad (26)$$

Substituting the values of Δ^{m+n} u_{00} in the equation (26) and using $T = \frac{t}{1/2}$, we get

$$u(x,t) = x - x(x-1) - xT$$

$$+\frac{x(x-1)T}{2} - \frac{xT(T-1)}{2}$$

$$\frac{-x(x-1)(x-2)T}{6} + \frac{x,T.(T-1)(T-2)}{6}$$

$$+\frac{1}{24}x(x-1)(x-2)(x-3)T$$

$$+\frac{1}{12}x(x-1)(x-2)T(t-1)$$

$$-\frac{1}{12}x(x-1)\mathrm{T}(x-1)(T-2)$$

$$-\frac{1}{24}xT(T-1)(T-2)(T-3) \qquad (27)$$

where T = 2t.

The equation (27) is the solution of (1) as a polynomial in x and t. It is observed that the values of u(x, t) computed at lattices points are tallied along and above the principal diagonal of the Table – I.

3. References:

1. J.B. Scarborough, Numerical Mathematical Analysis, Oxford and IBH publishing Co. Pvt. Ltd., (1966).
2. Stanley J. Farlow Partial differential equations for scientists and engineers, John Wiley & Sons, New York (1982) equations for scientists and engineer. John Wiley & Sons New York (1982)